On Teichmüller spaces for surfaces of infinite topological type

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In this work we studied Teichmüller spaces for surfaces of infinite topological type using geometric techniques, e.g. hyperbolic structures, pairs of pants decompositions and Fenchel-Nielsen coordinates. For the details you can see the original papers [1], [2], [3]. Here, by Teichmüller spaces we mean reduced Teichmüller spaces, a definition that is more suited to be studied using such techniques (as opposed to the techniques of complex analysis). (For the definition of the non-reduced spaces, see [6]).

In several ways, surfaces of infinite topological type are much more complicated than surfaces of finite type. They can't be classified by the genus and the number of punctures only, but there is a nice classification theorem, see [8]. The most important topological property we need is that every orientable surface of infinite type can be decomposed into pairs of pants, i.e. there exists a system of simple closed curves $(C_i) \subset S$ such that $S \setminus \bigcup_i C_i$ is a disjoint union of spheres minus three holes. This can be proved easily using the mentioned classification theorem.

To define Fenchel-Nielsen coordinates on Teichmüller spaces we have to show that every complex structure on S can be constructed by gluing hyperbolic pairs of pants. First we need to associate a hyperbolic metric to every complex structure, and for this we will use the **intrinsic metric**, defined by Bers; see [1] for details. Then, we need to characterize the hyperbolic metrics that can be constructed by gluing hyperbolic pairs of pants. Note that on surfaces of infinite type, it is not true that any topological pair of pants decomposition gives rise to a hyperbolic pairs of pants decomposition. The following theorem gives such a characterization.

Theorem 1. ([1]). *Let (S, h) be an orientable surface with a hyperbolic metric. The following are equivalent.*

(1) *(S, h) can be constructed by gluing hyperbolic pairs of pants.*
(2) *(S, h) is a convex core hyperbolic metric.*
(3) *For every topological pairs of pants decomposition $(C_i) \subset S$, there exists a pairs of pants decomposition $(\gamma_i) \subset S$ such that for all i, γ_i is a geodesic homotopic to C_i.*

Note that the intrinsic metric on every complex structure is always a convex core hyperbolic metric, hence this theorem can be applied to it. The implication (2) ⇒ (1) was proved in [4], but here we need (2) ⇒ (3), in order to define Fenchel-Nielsen coordinates on Teichmüller spaces.

Now let's discuss the definition of Teichmüller spaces in this context. Let Σ be a fixed orientable surface of infinite type. One would like to define the Teichmüller space of Σ in the following way:

$$\mathcal{T}(\Sigma) = \{(f, X) \mid X \text{ is a Riemann surface and } f : \Sigma \to X \text{ is a diffeo } \}/\sim$$

where $(f, X) \sim (f', X')$ if and only if there exists a biholomorphism $h : X \to X'$ such that $h \circ f$ is isotopic to f'. Note that this equivalence relation is the one giving the reduced theory of Teichmüller spaces; it is the most natural definition, but it is not the most widely used.

The set $\mathcal{T}(\Sigma)$ defined in this way parametrizes the complex structures on Σ, but it is not easy to find interesting structures on this set. To define distances, we need to consider some subsets of this set containing only comparable complex structures. To compare complex structures, we will use some functionals R defined over the diffeomorphisms $h : X \to Y$, and satisfying the following properties: $0 \leq R(h) \leq \infty$, $R(h) = 0 \iff h$ is a biholomorphism, and the triangle inequality $R(h \circ h') \leq R(h) + R(h')$. Given a functional h with these properties, one can define a distance in the following way:

$$d_R((f, X), (f', X')) = \inf_h R(h) \leq \infty$$

where the infimum is taken over all the h such that $h \circ f$ is isotopic to f'.

Then we need to choose a base point $\mathbb{X}_0 = (f_0, X_0)$, and we can define the following subset of $\mathcal{T}(\Sigma)$:

$$\mathcal{T}_R(\mathbb{X}_0) \quad = \quad \{(f, X) \mid d_R(\mathbb{X}_0, (f, X)) < \infty\}/\sim \quad \subset \quad \mathcal{T}(\Sigma)$$

The pair $(\mathcal{T}_R(\mathbb{X}_0), d_R)$ is a metric space.

We will discuss different definitions of Teichmüller spaces, for different choices of R. The most important one is when R is the quasiconformal dilatation of h, $R(h) = qc(h) = \log(K(h))$, where K is the quasiconformal constant of h. This gives rise to what we call the **quasiconformal Teichmüller space**, denoted by $(\mathcal{T}_{qc}(\mathbb{X}_0), d_{qc})$, a complete metric space. Another possibility is to use the length-spectrum dilatation, $R(h) = ls(h)$, defined by the following formula, for a diffeomorphism $h : X \to Y$:

$$ls(h) = \sup_\alpha \left\{ \left| \log \frac{\ell_Y(h(\alpha))}{\ell_X(\alpha)} \right| \right\}$$

where the sup is over the set of all simple closed curves α. This gives rise to what we call the **length-spectrum Teichmüller space**, denoted by $(\mathcal{T}_{ls}(\mathbb{X}_0), d_{ls})$. It is also possible to use Fenchel-Nielsen coordinates to define R. If $h : X \to Y$ is a diffeomorphism, and $(C_i) \subset X$ is a pairs of pants decomposition, also $(h(C_i)) \subset Y$ is a pairs of pants decomposition, and we can compare the Fenchel-Nielsen coordinates in the following way:

$$R(h) = FN(h) = \sup_i \max \left(\left| \log \frac{\ell_Y(h(C_i))}{\ell_X(C_i)} \right|, |\tau_Y(h(C_i)) - \tau_X(C_i)| \right)$$

After having chosen a fixed pairs of pants decomposition $(C_i) \subset \Sigma$ of our base topological surface, this gives rise to what we call the **Fenchel-Nielsen Teichmüller space**, denoted by $(\mathcal{T}_{FN}(\mathbb{X}_0), d_{FN})$. This space depends on the chosen pairs of pants decomposition of Σ, but it has a clear structure, with explicit coordinates, and it is isometric to the sequence space ℓ^∞. We will use this space to describe the others.

To do this we need some hypotheses on the base point of the space. A Riemann surface X is **upper bounded** with reference to a pairs of pants decomposition

$(C_i) \subset X$, if there exists a constant M such that for all i we have $\ell(C_i) \leq M$. We have the following:

Theorem 2. (See [1]). *If* $\mathbb{X}_0$ *is upper bounded, then* $\mathcal{T}_{qc}(\mathbb{X}_0) = \mathcal{T}_{FN}(\mathbb{X}_0)$ *and the identity map* $id : (\mathcal{T}_{qc}(\mathbb{X}_0), d_{qc}) \to (\mathcal{T}_{FN}(\mathbb{X}_0), d_{FN})$ *is locally bi-Lipschitz. In particular* $\mathcal{T}_{qc}(\mathbb{X}_0)$ *is locally bi-Lipschitz equivalent to the sequence space* ℓ^∞.

The last remark should be compared with a recent result of A. Fletcher ([5]) giving a similar property for non-reduced Teichmüller spaces.

Now let's see some properties of the length-spectrum Teichmüller space.

Theorem 3. (See [2]). *The metric space* $(\mathcal{T}_{ls}(\mathbb{X}_0), d_{ls})$ *is complete.*

A very important fact is an inequality due to Wolpert (see [11]) that we can state as $d_{ls} \leq d_{qc}$. This also implies that $\mathcal{T}_{qc}(\mathbb{X}_0) \subset \mathcal{T}_{ls}(\mathbb{X}_0)$.

Shiga studied the length spectrum metric on the quasiconformal Teichmüller space and he introduced the following condition that we name after him: a Riemann surface X satisfies the **Shiga's condition** with reference to a pairs of pants decomposition $(C_i) \subset X$, if there exists a constant M such that for all i we have $\frac{1}{M} \leq \ell(C_i) \leq M$. Under this condition on the basepoint, he proved that d_{ls} and d_{qc} induce the same topology on $\mathcal{T}_{qc}(\mathbb{X}_0)$ (see [9]).

Later, Liu and Papadopoulos introduced the length-spectrum Teichmüller space, and they proved that under the same condition, the two spaces are the same set (see [7]). We refined these results as follows:

Theorem 4. (See [3]). *If* $\mathbb{X}_0$ *satisfies Shiga's condition, then* $\mathcal{T}_{ls}(\mathbb{X}_0) = \mathcal{T}_{qc}(\mathbb{X}_0) = \mathcal{T}_{FN}(\mathbb{X}_0)$ *and the identity maps between any two of these spaces are locally bi-Lipschitz, with reference to the respective distances* d_{ls}, d_{qc}, d_{FN}. *In particular* $\mathcal{T}_{ls}(\mathbb{X}_0)$ *is locally bi-Lipschitz equivalent to the sequence space* ℓ^∞.

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Domains of Discontinuity for Anosov Representations

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(joint work with Olivier Guichard)

1. Motivation

The concept of *Anosov representations* has been introduced by F. Labourie [8] in his study of Hitchin representations of surface groups. Anosov representations $\rho : \Gamma \to G$ can be defined for any word-hyperbolic group Γ into any semisimple (real) Lie group (see Definition 1 below). When Γ is a free group or a surface group, Anosov representations should be thought of providing generalizations of quasi-Fuchsian representations. The goal of this talk is to describe a geometric picture for Anosov representation similar to the following classical examples.

(1) **Teichmüller space:** Let S be a closed surface. The Teichmüller space $\mathcal{T}(S)$ can be realized as a connected component in the representation variety $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ consisting of discrete embeddings. Any such representation gives rise to an action of $\pi_1(S)$ on the hyperbolic plane $\mathbb{H}^2$, which is properly discontinuous, free and with compact quotient. The quotient is the surface S endowed with a hyperbolic structure.

(2) **Quasi-Fuchsian space:** Embedding $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PSL}(2,\mathbb{C})$ a neighborhood of $\mathcal{T}(S)$ is given by the space of Quasi-Fuchsian representations $\mathcal{QF}(S) \subset \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$. Every Quasi-Fuchsian representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ admits a ρ- equivariant embedding $\xi : S^1 \to \mathbb{CP}^1$. On the complement $\Omega = \mathbb{CP}^1 \backslash \xi(S^1)$, the action of $\pi_1(S)$ (via ρ) is properly discontinuous, free and with compact quotient. The quotient consists of two connected components, which are both surfaces homeomorphic to S, naturally endowed with a $\mathbb{CP}^1$-structure.

Examples of Anosov representations include so called higher Teichmüller spaces, i.e. Hitchin representations or positive representations into split real Lie groups (e.g. $\mathrm{SL}(n,\mathbb{R})$) and maximal representations into Lie groups of Hermitian type (e.g. $\mathrm{Sp}(2n,\mathbb{R})$), as well as their "Quasi-Fuchsian" deformations into complex Lie groups, [7, 4, 8, 3, 2].

The main result discussed here, is a construction of domains of discontinuity with compact quotient for all Anosov representations. As a consequence, we associate deformation space of geometric structures on compact manifolds to all higher Teichmüller spaces.